Journal of Pure and Applied Algebra 38 (1985) 159–166
North-Holland

ZERO DIVISORS IN ENVELOPING ALGEBRAS OF GRADED LIE ALGEBRAS

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Communicated by C. Löfwall
Received 19 January 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

Introduction

Let L be a graded Lie algebra over a field $\mathbf{k}$ of characteristic different from 2. If L is concentrated in even degrees, that is, if L is a Lie algebra in the ordinary sense, its enveloping algebra UL has no zero divisors: this is an easy consequence of the Poincaré–Birkhoff–Witt theorem. On the other hand, if L contains a non-zero element of odd degree x such that $[x,x]=0$, clearly UL has zero divisors.

Following R. Bøgvad [3], let us say that a graded Lie algebra is 'torsion-free' if $[x,x]\neq 0$ for every non-zero x of odd degree, and 'absolutely torsion-free' if it remains torsion-free after field extension to the algebraic closure $\bar{\mathbf{k}}$ of $\mathbf{k}$. Then our main result states:

Theorem. *The enveloping algebra UL of an absolutely torsion free graded Lie algebra L has no zero divisors.*

We also show examples of torsion-free Lie algebras over the reals whose enveloping algebras have zero-divisors.

The proof takes two steps: we first reduce to the case when L is concentrated in degrees 1 and 2 with $\dim L_1=\dim L_2=n<\infty$ (the '(n,n)-quadratic' case). Next we prove the theorem by induction on n, using suitable (non-commutative) localization and the fact, due to Bøgvad, that a solvable graded Lie algebra has finite global dimension if and only if it is absolutely torsion free.

Our interest in this question grew out from the analogy between regular sequences in commutative algebra and 'inert' sequences in Lie algebras developed in [1] and [5]. As a matter of fact, the reduction to the quadratic case (Proposition 2.1) was obtained by S. Halperin and the second author during the preparation of [5]. We also wish to acknowledge the kind cooperation of several geometers in Nice,

especially A. Galligo and M. Gaetano for assisting our first attempts to find counterexamples through MACSYMA, and Ph. Maisonobe for helpful suggestions.

When this paper was written, Clas Löfwall informed us that Bøgvad had obtained the main result some time before us: it is only because Rikard kindly allowed us to do so that we still publish our paper: since his proof is different - and quite short and elegant indeed - we could not do less than reproducing it here as an appendix.

Finally, since he is somehow responsible for what Rikard did, it is most appropriate to dedicate this paper to Jan-Erik Roos.

1. Quadratic Lie algebras

Let $\mathbf{k}$ denote a field characteristic different from 2, and $\bar{\mathbf{k}}$ be its algebraic closure. We assume the reader is familiar with the basic features of graded Lie algebras (e.g. [6, §5]). We shall denote by $\langle V \rangle$ the free Lie algebra generated by the graded vector space V, and by $\{V\}$ the abelian Lie algebra on V, namely V endowed with the null bracket.

$|x|$ will denote the degree of a non-zero (homogenous) element x.

Definition 1.1 (Bøgvad). A graded Lie algebra L is said to be *torsion-free* if $[x,x] \neq 0$ for any non-zero element x of odd degree in L; the algebra is *absolutely torsion-free* if $\bar{L} = L \otimes_{\mathbf{k}} \bar{\mathbf{k}}$ is torsion-free.

Definition 1.2. A graded Lie algebra L is *(n,p)-quadratic* if $L_i = 0$ for $i \neq 1,2$ and $\dim_{\mathbf{k}} L_1 = n, \dim_{\mathbf{k}} L_2 = p,\ n, p < \infty$.

Remark 1.3. A (n,p)-quadratic Lie algebra L is nothing but a quadratic map $q: L_1 \to L_2$, with $q(x) = [x,x]$: indeed $[x,y] = \frac{1}{2}\{q(x+y) - q(x) - q(y)\}$ as usual. Note that L_2 is included in the centre of L and the cocyle of the central extension

$$0 \to \{L_2\} \to L \to \{L_1\} \to 0$$

can be identified with $q \in \mathrm{Hom}(S^2 L_1, L_2) = H^2(L_1; L_2)$.

Proposition 1.4. *Let L be an absolutely torsion-free (n,p)-quadratic Lie algebra. Then there exists a basis $x_1, \ldots, x_n$ of L_1 such that $[x_1, x_1], \ldots, [x_n, x_n]$ are linearly independent in L_2 (and therefore $p \geq n$).*

Proof. Let us first prove that $p \geq n$. For this, we may assume that $\mathbf{k} = \bar{\mathbf{k}}$ without loss of generality. Consider the quadratic map $g: L_1 \to L_1 \otimes L_1$ defined by $g(x) = x \otimes x$, and the bracket map $[\cdot]$: $L_1 \otimes L_1 \to L_2$. The linear subspace $\mathrm{Ker}[\cdot]$ of $L_1 \otimes L_1$ has codimension less than p, and the affine cone $g(L_1)$ has dimension n. Therefore if they intersect in a single point, one must have $n \leq \mathrm{codim}\ \mathrm{Ker}[\cdot] \leq p$.

Now let $x_1, \ldots, x_r$ be a linearly independent sequence in L_1, such that $[x_1, x_1], \ldots, [x_r, x_r]$ are linearly independent, and assume $(x_1, \ldots, x_r)$ is maximal with respect to this property. Then if $r < n$, there exists x_{r+1} independent from $x_1, \ldots, x_r$, but $[x_{r+1} + \sum \lambda_i x_i, x_{r+1} + \sum \lambda_i x_i]$ must lie in the linear subspace X of L_2 spanned by $[x_1, x_1], \ldots, [x_r, x_r]$ for all (λ_i) in $\mathbf{k}^r$: applying this to those sequences (λ_i) with $\lambda_i = 0$ except for one or two indices, we obtain that $[x_{r+1}, x_{r+1}]$, $[x_{r+1}, x_i]$, and $[x_i, x_j]$ lie in X for all $i, j = 1, \ldots, r$. Thus the subalgebra generated by $x_1, \ldots, x_{r+1}$ is a $(r+1, r)$-quadratic Lie algebra. But this would contradict the first part of the argument, since clearly any subalgebra of an absolutely torsion-free Lie algebra is absolutely torsion-free. □

Remark 1.5. The above statement is false if L is torsion-free, but not absolutely so: indeed a $(n, 1)$-quadratic Lie algebra L is essentially a quadratic form on $\mathbf{k}^n$, and L is torsion-free iff this quadratic form defines 0: thus if $\mathbf{k} = \mathbb{R}$ and L is a $(n, 1)$-quadratic torsion-free Lie algebra, there exists a basis $x_1, \ldots, x_n$, y of L, with $|x_i| = 1$, $|y| = 2$, such that $[x_i, x_i] = y$, $[x_i, x_j] = 0$ for all $i, j = 1, \ldots, n$, $i \neq j$. A basis of UL_p for all $p \geq n$ is $(x_{i(1)} x_{i(2)} \cdots x_{i(k)} x_1^{p-k})$ where $(i(1), \ldots, i(k))$ runs through those sequences of integers such that $2 \leq i(1) < i(2) < \ldots < i(k) \leq n$, so that $\dim_{\mathbf{k}} L_p = 2^{n-1}$ for $p \geq n$. If UL has no zero divisors, this means that the product map $UL_p \otimes UL_q \to UL_{p+q}$ is non-singular: by Adam's Hopf invariant one Theorem, this may only occur for $n = 1, 2, 3, 4$. Indeed in the latter cases the product map is isomorphic to the multiplication of the real, complex, quaternions and Cayley numbers respectively: therefore UL has zero divisors iff $n > 4$.

Our next propostion shows that (n, n)-quadratic absolutely torsion-free Lie algebras constitute the hard-core of the situation.

Proposition 1.6. *Assume* $\mathbf{k}$ *is infinite. If* $p > n$, *any torsion-free* (n, p)*-quadratic Lie algebra is a central extension of a torsion-free* (n, n)*-quadratic Lie algebra by an abelian Lie algebra concentrated in degree* 2.

Proof. Clearly, if L is a quadratic algebra, any linear subspace of L_2 is a central ideal. By induction on $p - n$, we only need to prove that if $p > n$, there exists $v \neq 0$ in L_2 such that $L/(v)$ is torsion-free. But the affine cone $q(L_1)$ has dimension n in L_2, and since $\mathbf{k}$ is infinite we can choose v in the complement of $q(L_1)$ and one immediately checks that $L/(v)$ is torsion-free. □

We now proceed to study the global dimension of absolutely torsion-free quadratic Lie algebras.

Proposition 1.7. *Let* L *be a* (n, n)*-quadratic Lie algebra. Then if* L *is absolutely torsion-free, one has* $\mathrm{gl.dim}\, L = n$. *Moreover, the Yoneda algebra* $\mathrm{Ext}_{UL}^{**}(\mathbf{k}, \mathbf{k})$ *is generated by* $\mathrm{Ext}_{UL}^{1,1}(\mathbf{k}, \mathbf{k}) = L_1$, *and* $\dim \mathrm{Ext}_{UL}^{p,p}(\mathbf{k}, \mathbf{k}) = n!/p!(n-p)!$, $\dim \mathrm{Ext}_{UL}^{p,q}(\mathbf{k}, \mathbf{k}) = 0$ *if* $p \neq q$.

Proof. Since $\mathrm{Ext}_{UL}^{**}(\bar{\mathbf{k}},\bar{\mathbf{k}})=\bar{\mathbf{k}}\otimes_{\mathbf{k}}\mathrm{Ext}_{UL}^{**}(\mathbf{k},\mathbf{k})$, we may assume $\mathbf{k}=\bar{\mathbf{k}}$.

Now the Koszul construction $C^*(L)$ is the tensor product of the polynomial algebra on the suspension of the dual of L_1 and the exterior algebra on the suspension of the dual of L_2, and the differential is dual to the bracket map:

$$d={}^t[\cdot]:sL_2^{\cdot}\to S^2(sL_1^{\cdot}).$$

Let $y_1,\dots,y_n$ be a basis of L_2: the n quadratic equations $d(y_i')=0$, where (y_i') is the basis dual to (y_i), define 0 in the n-dimensional affine space L_1, therefore by standard intersection theory the ring $S(sL_1')/(d(sL_2'))$ is a complete intersection and the ideal $(d(sL_2'))$ is regular: thus the cohomology of the Koszul complex is:

$$H^*(C^*(L))=\mathrm{Ext}_{UL}^{**}(\mathbf{k},\mathbf{k})=S(sL_1')/(d(sL_2'))$$

and the latter is an artinian local $\mathbf{k}$-algebra whose Hilbert polynomial is

$$\sum_p \dim_{\mathbf{k}}\mathrm{Ext}_{UL}^{p,p}(\mathbf{k},\mathbf{k})\ t^p=(1+t)^n \qquad \square$$

We now give a proof of Bøgvad's theorem in the particular case we shall need, namely the case of quadratic Lie algebras.

Proposition 1.8. *Let L be a (n,p)-quadratic Lie algebra. Then L is absolutely torsion free iff* gl.dim L *is finite, and then* $\mathrm{gl.dim}\,L=\dim L_2=p$.

Proof. Again we may assume $\mathbf{k}$ algebraically closed: if L is not torsion-free, it contains an abelian subalgebra on a single generator of odd degree, which has infinite global dimension: therefore if L has finite global dimension, it is absolutely torsion-free. Conversely, if L is torsion-free, there is a central extension:

$$0\to\{V\}\to L\to L/(V)\to 0$$

where V is included in L_2 and $L/(V)$ is (n,n)-quadratic and absolutely torsion-free. By 1.7, one has $\mathrm{gl.dim}\,L/(V)=n$; on the other hand, one has $\mathrm{gl.dim}\{V\}=\dim_{\mathbf{k}} V=p-n$ since $\{V\}$ is evenly graded abelian. By the Hochschild–Serre spectral sequence, $\mathrm{gl.dim}\,L=\mathrm{gl.dim}\{V\}+\mathrm{gl.dim}\,L/(V)=p$.) $\square$

Remarks 1.9. If $\mathbf{k}=\mathbb{Q}$, a quadratic Lie algebra L of finite global dimension can be thought of as a special instance of a coformal elliptic space (with homotopy Euler characteristic $n-p<0$). S. Halperin's results [4] provide alternative proofs of 1.7 and 1.8 in this case. Note that if $n=p$, the space is both coformal and formal – in fact 'hyperformal'.

Quadratic Lie algebras also appear in the context of local algebra: if R is a noetherian local ring with residue field $\mathbf{k}$, the Yoneda algebra $\mathrm{Ext}_R(\mathbf{k},\mathbf{k})$ is the enveloping algebra of a graded Lie algebra and the latter is quadratic iff R is a complete intersection: see [8] for details.

Finally, for $n>1$ the classification of (n,n)-quadratic Lie algebras, generated in degree 1, is equivalent to classifying n-dimensional linear families of quadrics in projective $(n-1)$-space, and absolute torsion-freeness corresponds to the absence of base-point in the algebraic closure (for $n=1$ the classification is trivial: there only is the free Lie algebra on one generator, which is torsion-free); if $\mathbf{k}=\mathbb{C}$, there is a single isomorphism class of (2, 2)-quadratic torsion free Lie algebra (namely the product $\langle x_1\rangle \times \langle x_2\rangle$ of two free Lie algebras on a single generator of degree 1), which corresponds to a non-singular involution on the projective line (if $\mathbf{k}=\mathbb{R}$, there are two such Lie algebras, according to whether the double points of the involution are real or not). For $n=3$, $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, one can use C.T.C. Wall's classification of nets of conics [9]: using his notation, the only types without base-points are A,B,D,E, and one can easily check that over $\mathbb{C}$ any (3, 3)-quadratic torsion-free Lie algebra is isomorphic to either $L(b)$ for some b in $\mathbb{C}$ or to $\langle x_1\rangle \times\langle x_2\rangle\times\langle x_3\rangle$, where $L(b)$ has the following presentation:

$$L(b)=\langle x_1,x_2,x_3\rangle/([x_1,x_1]-[x_3,x_3],[x_1,x_2],[x_1,x_3]-[x_2,x_2]+b[x_1,x_1])$$

Moreover $L(b)=L(b')$ if $b^2=b'^2$, and $L(b)$ corresponds to Wall's type A (resp. B,D) for $b^2\neq 0,1$ (resp. $b^2=1$, $b^2=0$), while $\langle x_1\rangle \times\langle x_2\rangle\times\langle x_3\rangle$ corresponds to type E.

2. Proof of the theorem

From now on we assume that the ground field $\mathbf{k}$ is algebraically closed. We first show that our theorem is true if it is true for (n,n)-quadratic Lie algebras.

Proposition 2.1. *The following statements are equivalent:*

(i) *The enveloping algebra of a torsion-free Lie algebra has no zero divisors.*

(ii) *The enveloping algebra of a torsion-free Lie algebra concentrated in degrees* 1 *and* 2 *has no zero divisors.*

(iii) *The enveloping algebra of a torsion-free (n,n)-quadratic Lie algebra has no zero divisors.*

Proof. Clearly only the implications (iii) ⇒ (ii) ⇒ (i) require proofs.

(ii) ⇒ (i). Let L be a torsion-free Lie algebra. We define a filtration on UL by setting the odd (resp. even) degree components of L in filtration 1 (resp. 2). The (bi)graded algebra associated to this filtration is $E^0UL=UL^{\#}$, where the Lie algebra $L^{\#}$ is defined by $L_1^{\#}=L_{\mathrm{odd}}$, $L_2^{\#}=L_{\mathrm{even}}$, $L_i^{\#}=0$ for $i\neq 1,2$, and the restriction to $L_1^{\#}\otimes L_1^{\#}$ of the bracket of $L^{\#}$ is equal to the restriction to $L_{\mathrm{odd}}\otimes L_{\mathrm{odd}}$ of the bracket of L, and zero otherwise. Clearly $L^{\#}$ is torsion free iff L is so, and if E^0UL has no zero divisors, neither does UL.

(iii) ⇒ (ii). Let us first observe that a Lie algebra L is torsion-free (resp. UL has no zero divisors) iff this property holds for all subalgebras of finite type. Thus we can replace 'concentrated in degrees 1 and 2' by '(n,p)-quadratic' in (ii). Now by 1.6, any torsion-free (n,p)-quadratic Lie algebra L is a central extension of a torsion-free (n,n)-quadratic algebra by an abelian algebra $\{V\}$ concentrated in degree 2. Filtering by the powers of the ideal $\{V\}$ yields as associated graded algebra the direct product $\{V\}\times L/(V)$, whose enveloping algebra is the tensor product of the polynomial algebra on V and $U(L/V)$: if (iii) holds, the latter has no zero divisors, so neither does UL. □

Thus we are left to prove (iii): since (iii) holds for $n=1$ (the enveloping algebra of a free Lie algebra on one generator is the polynomial algebra on one generator, which has no zero divisors), we shall prove (iii) by induction on n. Let L be a torsion-free (n,n)-quadratic Lie algebra, and let $x_1,\dots,x_n$ be a basis of L_1 such that $[x_1,x_1],\dots,[x_n,x_n\}$ is a basis of L_2 (see 1.4). Let $L(n-1)$ be the subalgebra of L generated by $x_1,\dots,x_{n-1}$ and L_2: this is an ideal of L with quotient $\{\mathbf{k}\cdot x_n\}$, therefore every element of UL can be uniquely written $a+bx_n$, with a and b in $UL(n-1)$. By induction hypothesis, $UL(n-1)$ has no zero divisors. Assume that UL admits a zero divisor $z=P+Qx_n$, with P and Q in $UL(n-1)$; note that $Q\neq 0$. In order to derive a contradiction, we need to be able to invert Q: to this end we now recall the necessary notions of non-commutative localisation.

Definition 2.2. Let A be a ring and $S\subset A$ be a multiplicative subset of A. One says that S is a (left) *Ore system* if the following conditions hold:

(O1) $\forall a\in A$, $\forall s\in S$, $As\cap Sa\neq\{0\}$.

(O2) S contains no zero divisors.

Then one has (cf. e.g. [2, §1]):

Proposition 2.3. *Let S be a Ore system in the ring A. Then A admits a ring of left fractions $S^{-1}A$, and the natural map $i:A\to S^{-1}A$ is injective and flat.* □

Remark. A flat localization $S^{-1}A$ exists under a milder hypothesis than (O2), see [7,ChII,§1], but i is no longer injective.

We can now state:

Proposition 2.4. *Let L be a torsion-free (n,n)-quadratic Lie algebra and $z=P+Qx_n$ be a zero divisor in UL as above. Then the multiplicative subset S of UL generated by Q and L_2 is a Ore system in UL.*

Proof. Observe that the elements of S commute with one another since L_2 is in the centre of L, and S contains no zero divisors by induction hypothesis: thus condition

(O2) is satisfied. To prove (O1), assume there exists a in UL and s in S such that $UL\cdot s\cap S\cdot a=\{0\}$. Set $|a|=p$ and $|s|=q$. The Poincaré series of UL is $(1+t)^n/(1-t^2)^n=1/(1-t)^n$, while S contains UL_2 whose Poincaré series is $1/(1-t^2)^n$. The relation $UL\cdot s\cap S\cdot a=\{0\}$ implies the following coefficient-wise inequality between formal series

$$t^p/(1-t)^n+t^q/(1-t^2)^n \ll 1/(1-t)^n.$$

Comparing the coefficients of t^{j+p+q} yields

$$(j+q+n-1)!/(j+q)!+((j+p)/2+n-1)!((j+p)/2)!$$
$$\le (j+p+q+n-1)!/(j+p+q)!$$

for all j such that $j+p$ is even. When $j\to\infty$, the left-hand side is equivalent to $(1+1/2^{n-1})j^{n-1}$, while the right-hand side is equivalent to j^{n-1}, a contradiction. □

We now proceed to the homological part of the proof.

Lemma 2.5. *Let L and z as in* 2.4. *Let* $\bar{\mathbf{z}}=Q^{-1}z\in S^{-1}UL$. *Then the multiplication by* $\bar{\mathbf{z}}$ *is an acyclic differential on* $S^{-1}UL$.

Proof. Since i: $UL\to S^{-1}UL$ is injective, we shall identify UL with $i(UL)$. Next $[UL(n-1),x_n]\subset UL(n-1)$, therefore if $B\in UL(n-1)$, there exists $C\in UL(n-1)$ such that $x_nB=Bx_n+C$. Now let $z'=P'+x_nQ'$ be an element in UL such that $zz'=0$. We can write:

$$(P+Qx_n)(P'+x_nQ')=0\Rightarrow (QP'-PQ')x_n=0\Rightarrow P'=Q^{-1}PQ'.$$

Hence $z'=(Q^{-1}P+x_n)Q'=\bar{\mathbf{z}}Q'$, and $\bar{\mathbf{z}}z'=\bar{\mathbf{z}}\bar{\mathbf{z}}Q'=0\Rightarrow \bar{\mathbf{z}}^2=0$. □

Let Ω be the exterior algebra $\mathbf{k}\oplus\mathbf{k}\cdot\bar{\mathbf{z}}$; by 2.5, Ω is a subalgebra of $S^{-1}UL$, and $\mathrm{Ext}^m_\Omega(\mathbf{k},S^{-1}UL)=0$ for $m>0$.

Lemma 2.6. *Let M be a* $S^{-1}UL$*-module. One has* $\mathrm{Ext}^m_\Omega(\mathbf{k},M)=0$ *for* $m>0$.

Proof. Let us consider the change of rings spectral sequence:

$$E_2^{p,q}=\mathrm{Ext}^p_{S^{-1}UL}(\mathrm{Ext}^q_\Omega(\mathbf{k},S^{-1}UL),M)\Rightarrow \mathrm{Ext}^{p+q}_\Omega(\mathbf{k},M).$$

By 2.5, this spectral sequence is trivial and $\mathrm{Ext}^*_\Omega(\mathbf{k},M)=\mathrm{Ext}^*_{S^{-1}UL}(\bar{\mathbf{z}}\cdot S^{-1}UL,M)$. Since $S^{-1}UL$ is UL-flat, $\mathrm{Ext}^*_{S^{-1}UL}(N,M)=\mathrm{Ext}^*_{UL}(N,M)$ for all $S^{-1}UL$-modules M and N. On the other hand, by 1.8 we know that $\mathrm{Ext}^m_{UL}(N,M)=0$ for $m>n$ since UL is torsion-free. Hence $\mathrm{Ext}^m_\Omega(\mathbf{k},M)=0$ for $m>n$. But $\mathrm{Ext}^m_\Omega(\mathbf{k},M)=\mathrm{Ker}\,\bar{\mathbf{z}}/\mathrm{Im}\,\bar{\mathbf{z}}$ is independent of m for $m>0$ and thus $\mathrm{Ext}^m_\Omega(\mathbf{k},M)=0$ for all m>0. □

We are now ready to end up our proof of the theorem:

Theorem 2.7. *Let L be an absolutely torsion-free Lie algebra. Then UL has no zero divisors.*

Proof. Let L and z be as above; let us consider $\bar{z}\cdot S^{-1}\cdot\bar{z}$, with the $S^{-1}\,UL$-structure defined by $\forall a,b\in S^{-1}\,UL,\ a\cdot(\bar{z}b\bar{z})=\bar{z}ab\bar{z}$; the action of $\bar{z}$ is trivial, therefore $\bar{z}\cdot S^{-1}\,UL\cdot\bar{z}=0$ by 2.6. But this means that $S^{-1}\,UL\cdot\bar{z}\subset\bar{z}\cdot S^{-1}\,UL$ by 2.5, and thus that the multiplication by $\bar{z}$ is trivial on the left ideal $S^{-1}\,UL\cdot\bar{z}$: again by 2.6 we get $S^{-1}\,UL\cdot\bar{z}=0$, i.e. $\bar{z}=0$, the contradiction we sought for. □

We conclude this paper with R. Bøgvad's proof: let L be any torsion-free quadratic Lie algebra over an algebraically closed field $\mathbf{k}$. Let a, b be in UL with $a\cdot b=0$, let $M=UL/UL\cdot a$, and let E_* be a finite UL-free resolution of M, which does exist since gl.dim $UL<\infty$. Let Φ be the fraction field of the polynomial algebra UL_2: since UL is a free UL_2-module (side is irrelevant since UL_2 is central), no element of UL_2 is a zero divisor in UL, and $E_*\otimes_{UL_2}\Phi$ is a $UL\otimes_{UL_2}\Phi$-free resolution of $M\otimes_{UL_2}\Phi$. Hence $\chi_\Phi(E_*)=\chi_{UL}(E_*)\cdot\dim_\Phi(UL\otimes_{UL_2}\Phi)=\dim_\Phi(M\otimes_{UL_2}\Phi)$, where χ denotes the Euler characteristic. But $\dim_\Phi(M\otimes_{UL_2}\Phi)\geq\dim_\Phi(UL\otimes_{UL_2}\Phi)$, therefore $\chi_{UL}(E_*)=0$ or 1.

If $\chi_{UL}(E_*)=0$, then $\dim_\Phi(M\otimes_{UL_2}\Phi)=0$, which means that there exists $s\in UL_2\cap UL\cdot a$, $s\neq 0$: then $s\cdot b=0$, hence $b=0$.

If $\chi_{UL}(E_*)=1$, then $\dim_\Phi(UL\otimes_{UL_2}\Phi)=\dim_\Phi(M\otimes_{UL_2}\Phi)$, therefore $UL\cdot a\otimes_{UL_2}\Phi=0$: there must exist $s\in UL_2$ with $s\cdot a=0$, hence $a=0$. □

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